Solow Growth Model

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1 Key Predictions

- Vast income difference across countries cannot be fully explained by differences in inputs of capitals, at least quantitatively.
- Saving rate and technology are both exogeneous, while both turn out to be really important to growth.

2 Key Assumptions

- $Y = F(K, AL)$, where A is the effective labor unit, a measure of labor productivity, can be also called labor-augmenting technology. The specification has the desirable property Herrod-neutrality, that allows the model to converge to a steady-state capital to output ratio $\frac{K}{Y}$. This matches better with the fact that there is no trended change in the ratio over a long period of time. Alternatively, it is called Hicks neutrality if $Y = AF(K, L)$. The reason why it is called Hicks neutral is that the cost-minimizing factor inputs ratio $\frac{K}{L}$ is a constant under different scales of production target. In another specification, the technology can be assumed to be capital-augmenting, which is $Y = F(AK, L)$. This is some times called Solow neutral.
- CRS constant return of scale. $F(\lambda K, \lambda AL) = \lambda F(K, AL)$. The CRS assumption has a wide range of important implications. First of all, the production capacity is not constrained by a limited supply of natural resources, i.e. population, lands, and physical space. Second, there is sufficient market competition so that there are not many freely profitable opportunities that can be easily exploited by firms. Third, the marginal cost of production is constant. The firm cannot simply push the marginal cost all the way down by producing more and more products to earn infinitely positive profits. Fourth, the CRS assumption allows the nice property as described by Euler Theorem, i.e., the factor income is a constant share of total output in perfect competition.

There is another virtue of CRS. We can always normalize the aggregate production function with two inputs into a normalized version with only one input. Most commonly, we have $F(K, AL) = ALF(\frac{K}{AL}, 1)$, by defining $\frac{K}{AL} = k$ as the per effective labor unit. Now we have $F(K, 1) = f(k)$, the dynamics of the economy can be fully captured by the evolution of capital per effective labor unit. This is as if we equally divide the total capital K into AL islands and each island gets capital k. For CRS, the economies should produce exactly the same amount as the original economy.

Lastly, with CRS production the elasticity of substitution of different factors is equal to one. The factor demand increases by one-to-one manner.

3 Model Specifications

- Cobb-Douglass production function. $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$. Notice with the C-D form, labor-augmenting, capital-augmenting and Hicks neatural becomes the same.

$$\underbrace{F(K, AL)}_{\text{Labor augmenting}} = K^{\alpha}(AL)^{1-\alpha} = \underbrace{A^{1-\alpha}K^{\alpha}L^{1-\alpha}}_{\text{Hicks neatural}} = A(\frac{K}{A})^{\alpha}L^{1-\alpha} = \underbrace{A(A_{inv}K)^{\alpha}L^{1-\alpha}}_{\text{Capital augmenting}}$$

- Continuous time. In the continuous-time setting, the growth rate of some variables can be either understood as the first order derivative with respect to time t divided by its current level, or its log growth. For instance, see the following law of motion of aggregate capital K:

$$g_K = \frac{\dot{K}}{K_t} = \frac{\partial K_t}{\partial t}\frac{1}{K_t} \quad \Longrightarrow \frac{\partial lnK}{\partial t} = \underbrace{\frac{\partial lnK_t}{\partial K_t}\frac{\partial K_t}{\partial t}}_{\text{By Chain Rule}} = \frac{1}{K_t}\dot{K} = \frac{g_K K_t}{K_t} = g_K$$

Also, we know the growth rate of a variable, then its future level a time interval of Δt from now is

$$K_{t+\Delta t} = K_t e^{\Delta t g_K}$$

To see why taking log on both sides gives

$$ln(K_{t+\Delta t}) = ln(K_t) + \Delta t g_K \quad \Longleftrightarrow \quad ln(K_{t+\Delta t}) - ln(K_t) = \Delta t \frac{\dot{K}}{K_t}$$

4 Equilibrium and Dynamics of the Model

- Steady State. From an aggregate perspective, labor force and productivity growth expand the total production over time. In order to achieve a balanced growth where total output Y grows by a constant rate, it requires that the total capital accumulation K matches with the growth of AL. From per effective labor perspective, this means the per effective labor capital stays constant. Two forces prevent this from happening. First, population growth and technology growth makes the denominator grow over time, thus diluting capital assigned to each unit of effective labor. Second, capital depreciates. In order to keep k constant, the new accumulation capital $sf(k)$ needs to compensate for the dilution of k. This pins down the steady-state capital stock per effective labor.

  $$sf(k) = (n + g + \delta)k \Rightarrow k^* = \frac{s}{n + g + \delta}^{1/(1-\alpha)}$$

  Notice the steady-state capital per effective level is only determined by parameters independent from the initial level of capital stock, labor force, and productivity, etc. It only has to do with the growth rate of different factors, the share of capital in the production function, saving rate and depreciation.

  Lower saving rate s, faster labor force growth n, technology growth g and depreciation δ all lower steady-state capital stock. Also, with smaller the importance of capital in the production function, measured by α, k^* becomes smaller.

  Steady-state production converges to a constant level and increases with α and saving rate.

  $$y^* = (\frac{s}{n + g + \delta})^{\alpha/1-\alpha}$$

  As the per effective labor output is constant in steady-state, the total output just scales up by the growth rate of labor force and technology.

  $$ln(Y_t) = \alpha ln K_t + (1 - \alpha)(A_t + L_t)$$

  $$\frac{\dot{Y}}{Y_t} = \frac{\partial ln Y_t}{\partial t} = \alpha \frac{ln K_t}{\partial t} + (1-\alpha)\frac{\partial ln A_t}{\partial t} + (1-\alpha)\frac{\partial ln L_t}{\partial t} = (\alpha+1-\alpha)(g+n) = g+n$$

- Consumption in Steady State. Although the saving rate effect of output per effective worker unambiguously increases with saving rate, this is not the case for consumption.

$$\chi^* = \frac{C_t^*}{A_t L_t} = y_t^*(1-s) = k_t^{*\alpha}(1-s) = (1-s)(\frac{s}{n+g+\delta})^{\alpha/1-\alpha}$$

Notice now, on one hand, a higher saving rate allows the economy to produce more, but at the same time, it means a smaller fraction of income going to consumption. Let me call the first Size-of-Pie effect and the second Fraction-of-Pie effect. The real effect on consumption depends upon the counterbalancing of the two. In one extreme case, the stock of capital is far below k^* where the marginal production of capital is high. A marginal increase in saving rate results in a substantially bigger size of the pie. Thus the size effect dominates, thus consumption increases with the big pie. In the other extreme where k is significantly higher than k^*, the Fraction-of-Pie effect dominates.

The saving rate that maximizes χ is called Golden Rule(GR now after). As the Solow model just takes the saving rate as exogenously given, there is no guarantee that the saving rate is at the GR level.

Consumption is the total production left after break-even capital.

$$\chi = k^\alpha - (n+g+\delta)k$$

The first-order condition with respect to k

$$f'(k_{GR}) = n+g+\delta$$

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The GR capital level is such that its MPK equal to the growth rate of population, technological growth, and depreciation.

It is also interesting to see what the saving rate is under the Gold Rule. With the Cobb-Douglass production form, one can solve that the GR saving rate turns out to be equal to α, capital share.

$$\alpha k_{GR}^{\alpha-1} = n+g+\delta$$

$$k_{GR} = (\frac{\alpha}{n+g+\delta})^{\frac{1}{1-\alpha}}$$

Compare it with the steady-state k^* with any exogenous saving rate s, it clear that

$$s_{GR} = \alpha$$

Mathematically, this is equivalent to solving optimal s under steady-state.

Taking first-order derivative of the following equation:

$$\chi = k^{*\alpha}(1-s)$$

$$\Rightarrow \frac{\partial k^{*\alpha}}{\partial s_{GR}}(1-s_{GR}) = \alpha k^{*\alpha-1}\frac{\partial k^{*}}{\partial s_{GR}} = k^{*\alpha} \Rightarrow s_{GR} = \alpha^{1}$$

Intuitively, this implies that the GR saving rate is such that saving in each period exactly matches the share of capital in production. This makes the Size and Fraction effect break even.

5 Extensions

- Growth with Bounded Resources. The benchmark model above assumes the production capacity is not constrained by the limit of natural recourses such as land and environments. The boundedness of production factors can be modeled by assuming zero or negative growth rate of these factors over time. Consider the following example:

  $$Y_t = K_t^{\alpha} L^{\beta} R_t^{\theta} (A_t L_t)^{1-\alpha-\beta-\theta}$$

  where L is constant over time. And the growth rate of resource $\frac{\dot{R}}{R_t} = -b$ where $b > 0$.

  Now the economy cannot achieve a balanced growth rate with a constant level of capital per effective labor anymore. When we normalize the production function by A_tL_t, it is easily seen that the numerator and denominator are to the power of α and $\alpha+\beta+\theta$, respectively. If both bases grow by constant rates, the ratio of the two cannot stay the same.

  From an aggregate perspective, now the question becomes that there are certain factors that stay the same or decline over time. Since in such a production function every factor is a Q-complements, the lack of growth of certain factors will undoubtedly drag down the growth of the economy as a whole. It is then now clear that the counterbalancing occurs for technology and labor force growth against the bounded resources. Depending on the former or the latter dominates, the total output can grow at a positive, zero or negative rate. To see this:

  $$ln(Y_t) = \alpha ln(K_t) + \beta ln(L) + \theta ln R_t + (1-\alpha-\beta-\theta)A_tL_t$$

  Take the derivative with respect to time t, it gives

[1] $(\frac{s_{GR}}{n+g+\delta})^{\alpha/(1-\alpha)} = (1-s_{GR})(\frac{1}{n+g+\delta})^{\alpha/(1-\alpha)}\frac{\alpha}{1-\alpha}s_{GR}^{(2\alpha-1)/(1-\alpha)} \Rightarrow s_{GR}^{\alpha/(1-\alpha)} = (1-s_{GR})\frac{\alpha}{1-\alpha}s_{GR}^{(2\alpha-1)/(1-\alpha)} \Rightarrow s_{GR} = \alpha$

$$\frac{\partial lnY_t}{\partial t} = \alpha \frac{\partial lnK_t}{\partial t} - \theta b + (1-\alpha-\beta-\theta)(n+g)$$

Combining

$$\frac{\dot{K}}{K_t} = \frac{sY_t}{K_t} - \delta$$

We have

$$\frac{\partial lnY_t}{\partial t} = \alpha(\frac{sY_t}{K_t} - \delta) - \theta b + (1-\alpha-\beta-\theta)(n+g)$$

Now it is clear that for the output to grow at a constant rate, $\frac{Y_t}{K_t}$ needs to be a constant, which implies Y_t and K_t grow at the same rates. Call it γ, then we can solve it as below and it can be positive, zero or negative.

$$\gamma = \frac{(1-\alpha-\beta-\theta)(n+g) - \theta b - \alpha\delta}{1-\alpha}$$

This seems to be assured that even with limited natural resources, an economy can still grow at a positive rate as long as the technology grows fast enough.

But don't rush to the conclusion. Let's consider what happened to income per capita. it should grow at the following rate, which can be also positive or negative.

$\gamma - n = \frac{(1-\alpha-\beta-\theta)(n+g)-\theta b-\alpha\delta-n-n\alpha}{1-\alpha} = \frac{(1-\alpha-\beta-\theta)g-\theta b-(2n+\beta+\theta+\delta)\alpha}{1-\alpha}$

In summary, in presence of bounded resources, there is no way for the economy to achieve a steady-state per effective labor income, but positive growth of total output and income per capita is still possible.

6 Quantitative Implications

- Cross-country difference in income per capita. Now back to the baseline Solow model. To what extent the cross-country difference of income can be explained using this model? The simple answer is no.

  We have derived income per capita in steady-state as below

$$\frac{Y_t}{L_t} = k^{*\alpha} A_t = (\frac{s}{n+g+\delta})^{\alpha/1-\alpha} A_t$$

  Take everything as given, a difference of income per capita ratio of 2 is corresponding to a ratio of saving rates equal to $2^{\alpha/(1-\alpha)}$. Assuming $\alpha = 0.3$, then the ratio of saving rate is approximately 1.4, or 40% increase in saving rate. This is not consistent with empirical facts of saving rate and income per capita. To account for the vast income difference of different countries, one need unrealistic differences in saving rates across countries.

Another way to look into this is to explicitly work out the elasticity of income per capita with respect to saving rate. And the same messages emerge.

$$\frac{\partial ln(Y/L)}{\partial s} = \frac{\alpha}{1-\alpha} \approx 0.5$$

- Convergence Speed.
- Growth Accounting.

7 Discussions

The key feature in the model is that in steady-state, there is no perpetual growth in per effective labor.

One natural extension of the Solow model is to endogenize the saving rate by deriving it from a consumer's optimization problem. One would expect that once the saving decision becomes an inter-temporal choice, the saving rate is going to be also dependent upon the preferences of economic agents. Most importantly, the discount factor or patience.

Further extension is to allow for the role of different types of agents to make the saving decisions. This allows the economy deviates from the first best due to the decentralized agents are not infinitely long-lived.