Overlapping Generation Models

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1 Key Predictions

- **Dynamic Inefficieny**. Decentralized decision making by individual households with finite life horizon leads to higher savings than the socially optimal level of capital stock. On one hand, the socially optimal level of capital is such that per capita consumption is maximized for each generation. It is when the marginal product of capital is equal to the speed of dilution of the capital from population growth and depreciation. On the other hand, individual consumption and saving are determined where the marginal product of capital is equal to time preference. If the first is higher than the second, the implied capital stock by the first is lower than the second. This implies there is over saving and dynamically inefficiency.

2 Model

2.1 Setting

- **Households**. Two generations denoted by supscripts $\{1, 2\}$. Two periods t and $t+1$. For instance, C_t^1 denotes the total consumption of young generation in period t. c_t^1 denotes the consumption of young generation per capita. $C_t^1 = N_t c_t^1$.

- **Households**. Young people earn labor income from the production sector in the same period, which uses the capital saved by the contemporaneous old generations. Young people decide the amount to consume and save given the next period marginal production of capital, namely real interest rate. Old people earn capital income from their period t saving and spend all. A classical Fisher two-period intertemporal problem. The Euler equation under the CRRA utility function is standard as below.

$$\frac{c_t^1}{c_{t+1}^2} = (\beta R_{t+1})^{1/\rho}$$

  Combining budget constraints $c_t^1 + \frac{c_{t+1}^2}{R_{t+1}} = W_t$ gives the level of consumption and saving by the young people.

$$c_t^1 = \frac{1}{1 + (\beta^{1/\rho} R_{t+1}^{(1-\rho)/\rho})} W_t$$

$$K_{t+1} = N_t \frac{(\beta^{1/\rho} R_{t+1}^{(1-\rho)/\rho})}{1 + (\beta^{1/\rho} R_{t+1}^{(1-\rho)/\rho})} W_t = N_t s_t(R_{t+1}) W_t$$

The total saving of the economy in $t+1$ is equal to the aggregate capital stock of capital K_{t+1}. And s_t is the saving rate, as a function of R_{t+1}.

With postive labor growth rate n between t and $t+1$ and deviding the equaction by N_{t+1}, capital per capita in $t+1$ is accumulated from t through new saving net of the dillusion of population.

$$k_{t+1} = \frac{1}{(1+n)} s_t W_t$$

- **Firms.** Perfect competition on the production side. Both wages and return of capital are determined by their marginal production respectively.

$$R_t - 1 = f'(k_t)$$

$$R_{t+1} - 1 = f'(k_{t+1})$$

$$W_t = f(k_t) - k_t f'(k_t)$$

Under Cobb-Douglass production function where wage and capital income are constant shares of the total production.

$$W_t = (1-\alpha) f(k_t)$$

2.2 The Law of Motion of Capital and Steady State

Saving by the young is a fraction of the wage, and the wage is a function of aggregate capital saved by the previous generation. In the meantime, the current saving adds to the change in the capital stock net of dilusion. This is described by the following law of motion between k_t and k_{t+1}

$$k_{t+1} = \frac{1}{1+n} s_t \underbrace{(R_{t+1})}_{\text{Function of } k_{t+1}} (1-\alpha) f(k_t)$$

$$k_{t+1} = \frac{1}{1+n} s_t(k_{t+1})(1-\alpha) f(k_t)$$

The state variable of the economy is capital per capita k. In order to find the steady-state, we can graphically find the intersection of the above equation and 45-degree line. That is where $k_t = k_{t+1} = k^*$.

Up till now, the exact shape of $k_{t+1}(k_t)$ is not determined. We only know that it is an upward curve as $(1-\alpha) > 0$. But there is no guarantee that the slope of the curve is monotonic such that its fixed point is uniquely determined. In general, there could be multiple-equilibria. Or if the initial capital stock is too low, there might be no equilibrium.

2.3 Special Case of Log Utility

With log utility, saving rate is no longer a function of k_{t+1}. The substitution effect and income effect cancel out. As there is no human wealth effect in this context, the consumption and saving decision is exogenously determined by patience factor only.

$$s = \frac{\beta}{1+\beta}$$

$$k_{t+1} = \frac{1}{1+n} s(1-\alpha) f(k_t)$$

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+n} s(1-\alpha) f'(k_t)$$

At a small value of k^*, $f'(k_t)$ is big, thus the slope of the curve is greater than one. As k_t grows, the slope gradually becomes smaller than one. This is how we have a unique steady-state capital k^*

$$k^* = \frac{1}{1+n} s(1-\alpha) k^{*\alpha}$$

$$k^* = (\frac{s(1-\alpha)}{1+n})^{\frac{1}{1-\alpha}}$$

$$y^* = (\frac{s(1-\alpha)}{1+n})^{\frac{\alpha}{1-\alpha}}$$

$$f'(k^*) = R^* - 1 = \alpha k^{*(\alpha-1)} = \frac{\alpha}{(1-\alpha)} \frac{1+n}{s}$$

In the steady-state, capital per capita and output per capita remain unchanged. The economy grows by the rate of population. Not just in this special case of log utility, the steady-state saving rate is constant since the capital stock is unchanged. These features are the same as in the Ramsey and Solow model.

Higher saving rate, resulting from a higher discount rate β will increase the steady-state capital.

Faster growth of the labor force, namely more young people joining the production compared to the old people who provide the capital, implies lower steady-state capital.

A higher share of capital in the production, a bigger value of α, results in a higher steady-state capital.

It is also interesting to look at the consumption per capita in the steady-state. Higher capital share α and faster population growth n lowers young people's consumption. The effect of the saving rate is less obvious. On one hand, a higher saving rate implies a smaller proportion of income gets consumed. On the other hand, the higher saving rate allows the steady-state output to be higher.

$$c^{1*} = (1-s)(1-\alpha)f(k^*)$$

$$c^{1*} = (1-s)(1-\alpha)(\frac{s(1-\alpha)}{1+n})^{\frac{\alpha}{1-\alpha}}$$

$$c^{1*} = (1-\alpha)^{\frac{1}{1-\alpha}}(1-s)(\frac{s}{1+n})^{\frac{\alpha}{1-\alpha}}$$

The consumption by the old generation is the following. There are no savings by the old and they just send out all the capital income. It is multiplied by $1+n$ as there are more young people in the population. Higher capital share, faster population growth, the higher saving rate of the young all make the old consumption higher.

$$c^{*2} = (1+n)\alpha f(k^*)$$

Also, we can work out the average consumption of the total population in time t.

$$c^* = ((1-s)(1-\alpha) + (1+n)\alpha)f(k^*)$$

$$c^* = ((1-(1-\alpha)s + n\alpha)f(k^*)$$

3 Problem of Dynamic Inefficiency

Now let's consider the welfare implications. Since there are different generations in the model, a hypothetical social planner needs to allocate weights to current and future generations. This is done as if the social planer chooses a discount factor maximizing her expected utility over a period of time. The closer the discount factor is to one, the more concerned she is about the future generations. She may even have a discount factor greater than one if she cares about the total utility instead of in per capita terms with positive population growth.

Call the social discount factor η. The social planner maximizes her discount utility as the following.

$$u(c_0^2) + \sum_{t=0}^{T} \eta^T (\underbrace{u(c_t^1) + \beta u(c_{t+1}^2)}_{\text{Life-long utility of generation t}})$$

The constraints faced by the planner is the total capital accumulation equation.

$$K_{t+1} - K_t = N_t f(k_t) - N_{t-1} c_t^2 - N_t c_t^1$$

Dividing by N_t, gives the following.

$$(1+n)k_{t+1} - k_t = f(k_t) - \frac{1}{1+n} c_t^2 - c_t^1$$

With the intermediate steps omitted, the optimal condition for social planer is as below

$$f'(k_t) + 1 = \frac{\eta}{1+n}$$

The faster the growth rate of the population, the higher the capital required. The lower the discount rate of the social planer, the lower the capital is allocated by the social planer.

We can solve the socially optimal capital stock as below.

$$k_{SP} = \frac{\alpha}{\frac{\eta}{1+n} - 1}^{\frac{1}{1-\alpha}} \neq k^* = (\frac{s(1-\alpha)}{1+n})^{\frac{1}{1-\alpha}}$$

Compare it with the decentralized equilibrium capital k^*, there is no guarantee that the two are equal. This means in general, the social planer's optimal can not be achieved by decentralizing decisions.

Since the first best is not necessarily achievable, is there another weaker optimality condition that the decentralizing equilibrium can at least satisfy? For instance, is the equilibrium Pareto efficient?

Rewrite the dynamic constraint equation, where χ_t is the consumption per capita at time t.

$$(1+n)k_{t+1} - k_t = f(k_t) - \underbrace{\frac{1}{1+n} c_t^2 - c_t^1}_{\chi_t}$$

$$\chi_t = f(k_t) + k_t - (1+n)k_{t+1}$$

In steady state, $k_t = k_{t+1} = \bar{k}$

$$\bar{\chi} = f(\bar{k}) - n\bar{k}$$

The capital level that maximizes the consumption per capita, namely the Gold Rule capital level is given below. Again, it is when the marginal product of capital is equal to the speed of dilution of capital per worker. The same as in the Solow model.

$$f'(\bar{k}) = n$$

$$k_{GR} = (\frac{\alpha}{n})^{\frac{1}{1-\alpha}}$$

This level of capital turns out to be lower than the steady-state equilibrium in a decentralizing economy. By reducing the total capital of the economy $k*$ to k_{GR}, saving of the young, both young and old generation can have higher consumption per capita.

This is not only the same period but also in all future periods. Imagine starting period t, the social planner reduce total capital from k^* to k_{GR}. Then the consumption in period t is $f(k^*) + (k^* - k_{GR}) - nk_{GR}$. It is higher than $f(k^*) - nk^*$. In all future periods, k_{GR} in steady-state, per capita consumption is also maximized.

The over-accumulation of capital in steady-state than what is required for Pareto optimal is called dynamic inefficiency.