Aggregation in Macroeconomics

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1 Overview

Allowing for heterogeneity in state variables faced by individual agents populating in an economy poses a tractability challenge to macroeconomists. It typically makes it difficult to aggregate individual variables into tractable equations that characterize the macro dynamics. Examples of such can be seen everywhere. For instance, overlapping-generation models in which agents are at different stages of their life cycles. Sticky price models in which firms are unable to reset price instantaneously in each period. Sticky expectation models in which agents infrequently update their information about newly arrived shocks.

One of the commonly used techniques that tackle such a challenge is to assume the agents are faced with an independently distributed, memoryless, and non-state-dependent stochastic shock that governs transition across states. The shock is typically assumed to follow a point Poisson process. It turns out such an assumption brings about a number of convenient features that allow the elegant aggregation of decentralized dynamics. We discuss such a modeling technology in this note.

2 An example: the Cupid arrow

At any point of the time t, the probability of each individual running into his(her) love is at a rate of p. The "love" shock is independently distributed across agents. Also, regardless of the state the individual is in, the probability remains the same.[1].

A few immediate convenient facts can be derived.

First, the average time waiting before someone shows up is $1/p$. To see this, let us consider the time for waiting for t_W to be a random variable. The probability density of this random variable is:

[1]This also can be thought of as a special case of the survival/duration models, in which the hazard/failure rate is a constant, independent from the duration of survival. In general, in this kind of models, the failure rate $\frac{f(t_w)}{1-F(t_w)}$ could be dependent on t_w, namely the duration. Here, the $f(t_w)$ is the pdf for an eponential distribution pe^{-pt} and the $F(t_w) = 1 - e^{-pt}$ is the cdf of t_w from $-\infty$ to t_w. Hence the hazard rate is exactly p.

$$f(t_w) = pe^{-pt_w} \tag{1}$$

The above equation can be easiest understood in a discrete-time setting.

$$p(t_w) = (1-p)^{t_w} p \tag{2}$$

The first term $(1-p)_w^t$ is a multiplicative probability of non-encountering for t_w consecutive periods. The second term p is the probability of encountering in the last period. The continous-time counterpart of $(1-p)^{t_w}$ is exactly $e^{-p_w^t}$.

Some acute readers may find immediately this is the pdf of an exponential distribution. Therefore, the average waiting time is the expectation of an exponentially distributed random variable. If x follows an exponential distribution, $f(x) = \lambda e^{-\lambda x} \forall \lambda > 0$, then $E(x) = 1/\lambda$.

$$E(t_w) = \int_0^{+\infty} t_w p e^{-pt_w} dt_w = \frac{1}{p} \tag{3}$$

Intuitively speaking, if within each unit of time, something happens at a rate being p, then the average time before that event happens is $1/p$.

The second handy fact from the above assumptions is that at any point of the time, there is a constant fraction of p of the population who encounter someone. Or put it differently, if there are N individuals in the economy, as $N \to +\infty$, Np people fall in love at any point in time. This follows from the Law of Large Numbers.

To see this, define $\hat{N}$ as the number of people that encounter their love. It is a random variable. The probability density function of $\hat{N}$ is

$$p(\hat{N}) = p^{\bar{N}}(1-p)^{N-\bar{N}} \tag{4}$$

Mean of $\bar{N}$ can be simply computed as a Binomial distribution with parameter N and p.

$$E(\bar{N}) = \sum_0^N \bar{N} p^{\bar{N}} (1-p)^{N-\bar{N}} = Np \tag{5}$$

This type of practice is quite common in macroeconomic modeling. It is what mathematicians call the mean-field approach. Macro dynamics, as a consequence of aggregation from micro agents, are typically complicated due to individual heterogeneity and interactions among agents. But the mean-filed approach is to characterize these potentially complicated aggregated dynamics by focusing on the "average" behaviors of the environment. This approach salvages tractability and elegance to the liking of macroeconomists for a long time, but it circumvents many important dimensions of the inquiries where individual heterogeneity and interpersonal interactions are essential for macro outcomes thus cannot be simply ignored.

We survey a few concrete applications of such a technique in macroeconomic models.

3 Calvo Pricing (Calvo (1983))

There are many ways of modeling nominal rigidity in dynamic New Keynesian models. Calvo pricing represents one of the tractable approaches widely used in the literature, and also in some sense, inspired many similar applications in other macroeconomic models.

The nature of such a problem is that each of a continuum of firms facing monopolistic competitions needs to optimally set their own price $P_{i,t}$ taking aggregate price P_t as given but can only do so at the arrival of the "Calvo fairy", which has a constant probability θ in each period, independent of time elapsed and the states of the economy. Otherwise, the firm has to keep the price level $P_{i,t-k}$ they set previously in their latest resetting, say $t-k$.

The expected path of the aggregate price P_t, relevant to the individual optimal behaviors, could have been a complicated distributional object over all firms with different prices and different adjustment histories. But the Calvo assumption plus two additional convenient features of the NK model, as detailed below, make the aggregate price P_t easily described as a recursive equation as follows.

$$P_t = \left[\underbrace{\int_{S(t)} P_{t-1}(i)^{1-\varepsilon} di}_{\text{non-reseters price}} + \underbrace{(1-\theta)\left(P_t^*\right)^{1-\varepsilon}}_{\text{reseters' price}} \right]^{\frac{1}{1-\varepsilon}} \tag{6}$$

where we use $S(t)$ to denote the set of individual firms which are not able to reset price at time t, i.e. non-reseters.

Because of the CES price aggregator under *optimal* demand, we can first write the aggregate price as a quasi-linear form of individual prices aggregated over firms.

Because of the symmetry among firms in its optimal price, a $(1-\theta)$ fraction of the reseters firms all set the optimal price to be P^*.

Furthermore, because of the crucial assumption of time-independence and homogeneity in firms, the first term in the bracket can be replaced by $\theta P_{t-1}^{1-\varepsilon}$. Because "who" happens to be able to adjust does not depend on the price of that firm, the distribution of the prices across these reseters are the identical to that among *all* firms. This allows us to use aggregate price at $t-1$, P_{t-1}, times a reduced mass of θ to represent the average price indices of the non-reseters.

$$P_t = \left[\theta\left(P_{t-1}\right)^{1-\varepsilon} + (1-\theta)\left(P_t^*\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \tag{7}$$

A potentially confusing point regards the additional exponential term $1-\epsilon$ in the aggregation. Notice, even if we do not exactly linearly sum all prices over the distribution, the above logic still follows. To make it more transparent, let's relabel $\tilde{P}_{t-1}(i) \equiv P_{t-1}(i)^{1-\epsilon}$. Then we have

$$
\begin{aligned}
\int_{S(t)} P_{t-1}(i)^{1-\varepsilon} di &= \underbrace{\int_{S(t)} \tilde{P}_{t-1}(i) di}_{\text{relabeling}} \\
&= \underbrace{\theta \int \tilde{P}_{t-1}(i) di}_{\text{distribution remains the same}} \\
&= \theta(\left[\int P_{t-1}(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}})^{1-\epsilon} \\
&= \theta \left(P_{t-1} \right)^{1-\varepsilon}
\end{aligned} \tag{8}
$$

Dividing equation 7 by P_{t-1} on both sides and use $\Pi_t \equiv= \frac{P_t}{P_{t-1}}$ to denote inflation, we can obtain the following.

$$
\Pi_t^{1-\varepsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon} \tag{9}
$$

Additional log-linearization around steady state $P^* = P_t = P_{t-1}$ and $\Pi == 1$, we get the following.

$$
\pi_t = p_t - p_{t-1} = (1-\theta)(p_t^* - p_{t-1}) \tag{10}
$$

With perfectly price adjustment or zero rigidity, $\theta = 0$, $P_t = P^*$, i.e. all firms optimally set their price at time t.

4 Perpetual Youth (Blanchard (1985))

In a canonical overlapping generation model (OLG), agents in the economy are distributed across different ages of the life cycle and the consumption and savings differ across age groups. The "representative agent" in general does not exist any more in that aggregate behaviors cannot be represented by an average household in the economy. Instead, an explicit aggregation from decentralized consumption/saving decisions of agents at different age is needed to characterize the macro behaviors of the economy.

This means that one need to tract the age distribution and the consumption/saving policy of each age group at each point of the time. This is not impossible computationally, but analytically it becomes highly tedious and non-interesting. The perpetual youth model was proposed to address this issue.

Instead of a finite life with a terminal date, the model assumes agents are ageless, living a perpetual life subject to a constant probability of death p at any point of the time. This is exactly a point Poisson process, as we assumed above. It follows that agents have an expected life horizon of $1/p$ at any point of the time.

Putting aside the unrealistic implication from such an assumption that, in theory, agent may live forever, even the assumption that death probability is independent of time the agent staying alive is undoubtedly unrealistic.

But such an assumption immediately brings about tractability in two aspects.

First, a time-invariant age distribution, according to which the fraction of agents born at time s and staying alive till t is $e^{-p(t-s)}$, exponentially decreasing over age $t-s$. Integrating from $s=\infty$ to t and assuming that the initial size of the new borns at each point of the time is always p, then the total mass of the population is exactly equal to 1. Notice that this is also equivalent to an integration over age from 0 to ∞ at time t (different age groups).

$$\int_{s=-\infty}^{t} pe^{-e(t-s)}ds = 1 \tag{11}$$

The second convenient implication is that the *propensity* to consume and save is no longer a function of age, which comes from the "perpetual" assumption plus the constant probability of death. This is intuitive because for agents who remain alive at different age (or equivalently, born at different points of the time before t), the forward-looking decision problem is intrinsically the same because they have exactly the same path of future.

It is important to note, however, that the two results alone are not enough to make it easy to characterize the macro dynamics with only a few aggregate equations. To see this clearly, write the aggregate consumption C_t as integration over individual consumption at different ages.

$$C_t = \int_{s=-\infty}^{t} c(s,t)pe^{-(t-s)}ds \overbrace{=}^{?} \Phi(W_t) = \Phi(V_t + H_t) \tag{12}$$

Our goal is to write aggregate consumption as a function of aggregate wealth W_t or non-human wealth V_t and H_t after this aggregation, i.e. a function $\Phi(.)$. But the human wealth and non-human wealth might differ across age groups, resulting from potentially complicated income structure. We use $w(s,t)$ to denote the wealth of the cohort s and assume, in general, the consumption as a function of wealth is $\phi(w(s,t))$. Then, the integration can be further written as follows.

$$C_t = \int_{s=-\infty}^{t} c(s,t)pe^{-(t-s)}ds = \int_{s=-\infty}^{t} \phi(w(s,t))pe^{-(t-s)}ds \tag{13}$$

It is obvious by now that for a general function ϕ, one cannot further rewrite the RHS to a function of W_t, unless a special case where it is linear!

Many form of market incompleteness such as borrowing constraints or uninsured idiosyncratic income risks, will immediately cause non-linearity of ϕ. But even absent of these, the uncertainty regarding death, breaks the market completeness. It is an idiosyncratic risk facing the agents in the model.

The trick used in perpetual youth model to salvage this market incompleteness, is to assume the death risk is insured by by perfectly competitive insurance companies, who pay a premium equal to p to each unit of the wealth held by those alive, in exchange for the right to collect the accidental bequest wealth in the case of their death. At any point of the time, p fraction of total wealth is paid out as premium and exactly p fraction of the total wealth are collected as accidental bequest, leading to zero profit of insurance companies. Agents will all willingly accept this offer because it cashes out the wealth that could have been left unused by them.

Although not discussed in the original model, a similar insurance is equivalent in effect: a benevolent government collects these bequests each period and redistribute them back to those alive proportional to their current wealth. This is feasible the wealth distribution of the dead is exactly the same as those alive.

Such an insurance mechanism also solves another subtle issue in aggregation that is induced by the accidental bequest. Although the consumption is only made by those alive, the wealth could come from those who have just accidentally died.

With this insurance, the market is complete and the agents at different age essentially consume a constant fraction of their total wealth. With the additional log utility assumption which makes consumption independent of real interest rate, we have the following linear consumption function (see Blanchard and Fischer (1989) for the detailed optimization problem).

$$c(s,t) = \phi(w(s,t)) = (\theta + p)w(s,t) \tag{14}$$

where θ is the discount rate.

And it immediately follows that the aggregate consumption is a linear function of the total wealth.

$$C_t = (\theta + p)W_t \tag{15}$$

5 Sticky Expectation

One of the key implications from the sticky expectation on aggregate consumption dynamics is the following equation, which explains why current aggregate consumption growth is correlated with lagged consumption growth. This fact is inconsistent with a benchmark model where agents instantaneously update the information and reacts to shocks optimally as a permanent-income consumer.

$$\Delta C_{t+1} = (1-\lambda)R\Delta C_t + \epsilon_{t+1} \tag{16}$$

The derivation of the equation from the very beginning is tedious. But a few steps are enough to show the gist of it.

We assume that consumers are distributed over the unit interval. Therefore, the population average is simply the integral.

$$C_{t+1} = \lambda \underbrace{C^r_{t+1}}_{\textit{Rational C at t}} + (1-\lambda) \underbrace{C_{\tau_t < t+1}}_{\textit{Non-updated C at t+1}} \tag{17}$$

The key step forward uses the assumption that the individuals that are able to update in the economy are randomly selected from the population from the previous period. Therefore, the average default consumption by the non-updated population should be equal to the average consumption of the whole population at time t. Namely

$$C_{\tau_t<t} = C_t$$

Then we can rewrite the equation above.

$$C_{t+1} = \lambda C^r_{t+1} + (1-\lambda)C_t \tag{18}$$

Subtracting C_t by both sides yields

$$\underbrace{\Delta C_{t+1}}_{C_{t+1}-C_t} = \lambda(C^r_{t+1} - C^r_t) + (1-\lambda)\underbrace{\Delta C_t}_{C_t - C_{t-1}} \tag{19}$$

Next, it is important to recognize the expression in the first parenthesis $C^r_{t+1} - C^r_t$ should only reflect newly arrived shocks at time t. In its classical form, where p_t is the permanent income, θ_t and ϕ_t are i.i.d. permanent income and transitory income shocks, respectively. The consumption function of a fully informed consumer is the following.

$$C^r_{t+1} = \underbrace{p_t}_{p_{t-1}+\theta_t} + \frac{r}{R}\phi_t$$

Therefore

$$C^r_{t+1} - C^r_t = (p_{t+1} - p_t) + \frac{r}{R}(\phi_t - \phi_{t-1}) = \theta_t + \frac{r}{R}\Delta\phi_t \tag{20}$$

θ_t and $\Delta\phi_t$ are both mean-zero i.i.d shock at time t. We can define the whole term as ϵ_t.

$$\lambda(C^r_{t+1} - C^r_t) \equiv \epsilon_t \sim N(0, \lambda^2\sigma^2_\theta + 2\lambda^2\sigma^2_\phi) \tag{21}$$

This loosely proves equation 16. It is loose because we do not prove that there is R in the original equation. But this is a trivial step.

References

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