Constant Elasticity of Substitution(CES) Demand System and Price Aggregator

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1 Basic Facts

A consumer derives her utility from a bundle of one unit continuum of goods with constant elasticity of substitution (CES). Index the good by $i \in [0,1]$. The elasticity of substitution is ϵ. The greater value it has, the more substitutable across individual goods. The function collapses to Cobb-Douglass form with constant return of scale form when $\epsilon \to 1$.

Consumption bundle C is defined as below.

$$C = [\int_0^1 c_i^{1-\frac{1}{\epsilon}} di]^{\frac{\epsilon}{\epsilon-1}}$$

Then there are two nice features of such a demand.

First, there is a composite aggregate price representation of the consumption bundle.

$$P = [\int_0^1 p_i^{1-\epsilon} di]^{\frac{1}{1-\epsilon}}$$

p_i is the price of good i.

Second, we can express demand for individual good i as a fraction of the total consumption bundle.

$$c_i = \frac{p_i^{-\epsilon}}{P^{1-\epsilon}} C$$

This convenient fact has been widely used in New Keynesian Models. It is due to Dixit and Stiglitz (1977)

2 Derivation

The consumer's problem:

$$Max \quad [\int_0^1 c_i^{1-\frac{1}{\epsilon}} di]^{\frac{\epsilon}{1-\epsilon}} \quad s.t. \int_0^1 p_i c_i di \leq 1 \tag{1}$$

With λ being the Lagrangian multiplier, F.O.C. is as below

$$\frac{\partial C}{\partial c_i} = \frac{\epsilon}{\epsilon - 1} C^{\frac{1}{\epsilon}} c_i^{-\frac{1}{\epsilon}} = \lambda p_i \tag{2}$$

$$c_i = (\frac{p_i}{p_j})^{-\epsilon} c_j \tag{3}$$

$$p_i c_i = (\frac{p_i}{p_j})^{-\epsilon} p_i c_j = p_i^{1-\epsilon} p_j^{\epsilon} c_j \tag{4}$$

Taking the integral over i

$$1 = p_j^{\epsilon} c_j \int_0^1 p_i^{1-\epsilon} di \tag{5}$$

Solving c_j

$$c_j = \frac{p_j^{-\epsilon}}{\int_0^1 p_i^{1-\epsilon} di} = \frac{p_j^{-\epsilon}}{P^{1-\epsilon}} \tag{6}$$

This proves the second part. (A special case when C=1).

From Equation (3) we know price P can be the expenditure of buying one unit of C, that is

$$C = [\int_0^1 c_i^{1-\frac{1}{\epsilon}} di]^{\frac{\epsilon}{\epsilon-1}} = [c_j^{\frac{\epsilon}{1-\epsilon}} p_j^{\epsilon-1} \int_0^1 p_i^{1-\epsilon} di]^{\frac{\epsilon}{\epsilon-1}} = c_j p_j^{\epsilon} [\int_0^1 p_i^{1-\epsilon} di]^{\frac{\epsilon}{\epsilon-1}} \tag{7}$$

$$c_j = C p_j^{-\epsilon} [\int_0^1 p_j^{1-\epsilon} di]^{\frac{\epsilon}{1-\epsilon}} \tag{8}$$

Multiplying p_j on both sides

$$c_j p_j = C p_j^{1-\epsilon} [\int_0^1 p_i^{1-\epsilon} di]^{\frac{\epsilon}{1-\epsilon}} \tag{9}$$

Taking integral over j

$$E = C \int_0^1 p_j^{1-\epsilon} dj [\int_0^1 p_i^{1-\epsilon} di]^{\frac{\epsilon}{1-\epsilon}} = C[\int_0^1 p_i^{1-\epsilon} di][\int_0^1 p_i^{1-\epsilon} di]^{\frac{\epsilon}{1-\epsilon}} = C[\int_0^1 p_i^{1-\epsilon} di]^{\frac{1}{1-\epsilon}} \tag{10}$$

Set $C = 1$, then we have

$$P = [\int_0^1 p_i^{1-\epsilon} di]^{\frac{1}{1-\epsilon}} \tag{11}$$

This proves the first part.

3 With relative taste weights

Here is a more generalized case with taste weights. By the same derivation, we can also show that if a preference weight of w_i is assigned to good i in the CES aggregator, namely

$$C = [\int_0^1 w_i c_i^{1-\frac{1}{\epsilon}} di]^{\frac{\epsilon}{\epsilon-1}}$$

the price index will be

$$P = [\int_0^1 w_i^{\epsilon} p_i^{1-\epsilon} di]^{\frac{1}{1-\epsilon}}$$

And the individual demand is

$$c_i = \frac{w_i^{\epsilon} p_i^{-\epsilon}}{P^{1-\epsilon}} C$$